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A GENERAL FRAMEWORK FOR LOCAL ERROR ESTIMATION APPLIED TO MATERIAL NONLINEAR PROBLEMS

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Abstract. *The study focuses on an adaptive finite element method for quasi time dependent problems in solid mechanics. Especially in the field of nonlinear materials problems, stable and robust finite element methods are needed to solve problems with prescribed error tolerance. Increasing but also decreasing plastic areas over the whole time range require adaptive techniques which minimize the calculation time. For this error indicators with computable bounds are necessary to control the adaptive finite element process.*

We begin by formulating the equations for the flow theory of Prandtl–Reuss*. Using the Maximum of Plastic Dissipation we end up in the well known primal and dual mixed formulation of the IBV problem of plasticity. These variational inequalities are the starting point for developing error estimators for standard* Bubnov–Galerkin *finite elements. A viscoplastic regularisation technique is used to transform the variational inequality into a nonlinear variational equality.*

Then a short review on classical error estimation techniques for PDEs will be given. Later on the well known global explicit residual based error estimator of Babuška & Rheinboldt *[2] will be extended into a general framework to estimate local quantities. This concept can be applied for a large range of* Galerkin*–type finite element problems. The achieved goal oriented error estimator will lead to an optimized mesh for a minimized error* $|e(\bar{x})|$ *of a local quantity in* $x = \bar{x}$*. The presented error estimation technique will be verified by a numerical example.*

1 INTRODUCTION

In the framework of time dependent or quasi time dependent problems in solid mechanics the computational efficiency requires an adaption of the time and space discretisation.

Therefore *a posteriori* error estimators have to be included in the adaptive finite element process to generate efficient and reliable or – in other words – optimal meshes. In a general setting we want to present error estimators for local quantities, e.g. for maximum strains or selected displacements. To derive such goal oriented estimates, we have to introduce a dual problem corresponding to the strong form of the underlying primal problem, Eriksson, Estep, Hansbo & Johnson [10], Becker & Rannacher [4], and Cirak & Ramm [6]. In the context of the flow theory of elastoplasticity this concept has been extensively elaborated by Suttmeier & Rannacher [19] and Cirak & Ramm [7]. On the basis of these contributions the objective of the present study is set once again the stage in a rather general framework allowing for extensions in different directions during the presentation. These will be the application for softening materials using a viscoplastic regularization and the evaluation of the dual problem by different techniques like local Neumann problems.

2 INITIAL BOUNDARY VALUE PROBLEM FOR PRANDTL–REUSS PLASTICITY

The equilibrium equations for a bounded domain $\Omega \in \mathbb{R}^n$ with prescribed displacements $\boldsymbol{u}_D$ on the Dirichlet boundary Γ_D and traction forces $\boldsymbol{g}$ on the Neumann boundary Γ_N are given by

$$\begin{array}{rcll} -\mathrm{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) & = & \boldsymbol{f} & \text{in } \Omega \\ \boldsymbol{u} & = & \boldsymbol{u}_D & \text{on } \Gamma_D \\ \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}) & = & \boldsymbol{g} & \text{on } \Gamma_N \end{array} \tag{1}$$

Assuming small strains, the strain tensor reads

$$\begin{array}{rcl} \dot{\boldsymbol{\varepsilon}} & = & \dot{\boldsymbol{\varepsilon}}^{el} + \dot{\boldsymbol{\varepsilon}}^{pl} \\ \boldsymbol{\varepsilon} & = & \nabla^{sym} \boldsymbol{u} \end{array} \tag{2}$$

Further on the free Helmholtz–energy Ψ and the internal variables $\boldsymbol{q} = (\boldsymbol{\varepsilon}^{pl}, \alpha)$ are given. Isotropic hardening is characterized by the parameter α.

$$\begin{array}{rcl} \Psi & = & \Psi(\boldsymbol{\varepsilon}, \boldsymbol{q}) = \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^{pl}, \alpha) \\ & = & \dfrac{1}{2}\, \kappa\, (\mathrm{tr}\, \boldsymbol{\varepsilon}^{el})^2 + \mu\, \mathrm{tr}\, (\boldsymbol{\varepsilon}^{el})^2 + \displaystyle\int_0^\alpha \phi(\alpha) \end{array} \tag{3}$$

The material parameter κ is the bulk modulus and μ the second Lamé constant. The given actual yield stress $\phi(\alpha) = \sigma_y + \bar{K}\alpha$ depends on isotropic hardening. Here σ_y is the one-dimensional yield stress and $\bar{K}$ the isotropic hardening modulus. A nonlinear relationship for the yield stress combined with kinematic hardening can be applied in a straightforward manner. Evaluating the CLAUSIUS–DUHEM inequality (second law of thermodynamics) we get the dissipation inequality

$$\mathcal{D}^{pl} = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{pl} - \phi\, \dot{\alpha} \geq 0 \tag{4}$$

with the yield function $\Phi(\boldsymbol{\sigma}, \phi) = \|\mathrm{dev}\, \boldsymbol{\sigma}\| - \phi(\alpha) \leq 0$. Using the *Postulate of Maximum Plastic Dissipation*

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^{pl} - \phi\, \dot{\alpha} \geq \boldsymbol{\tau} : \dot{\boldsymbol{\varepsilon}}^{pl} - p\, \dot{\alpha} \qquad \forall\, \boldsymbol{\tau} \in \mathcal{E}, \qquad \mathcal{E} = \{\boldsymbol{\tau} : \Phi(\boldsymbol{\tau}, p) \leq 0\} \tag{5}$$

which means that among all possibles stress states $\boldsymbol{\tau}$ satisfying the yield condition the stress state $\boldsymbol{\sigma}$ which maximizes $\mathcal{D}^{pl}(\boldsymbol{\tau}, p; \dot{\boldsymbol{\varepsilon}}^{pl}, \dot{\alpha})$ is the actual stress state. Transforming the constraint optimization problem, given as a minimization problem $-\mathcal{D}^{pl}(\boldsymbol{\tau}, p; \dot{\boldsymbol{\varepsilon}}^{pl}, \dot{\alpha})$ into a sequence of unconstrained problems by introducing a Lagrange multiplier γ we end up in the classical evolution equations, LUENBERGER [15], SIMO & HUGHES [20].

$$\begin{aligned}
\mathcal{L}^{pl}(\boldsymbol{\tau}, p, \dot{\gamma}; \dot{\boldsymbol{\varepsilon}}^{pl}, \dot{\alpha}) &= -\boldsymbol{\tau} : \dot{\boldsymbol{\varepsilon}}^{pl} + p\, \dot{\alpha} + \dot{\gamma}\Phi(\boldsymbol{\tau}, p) \\
\frac{\partial \mathcal{L}^{pl}(\boldsymbol{\tau}, p, \dot{\gamma}; \dot{\boldsymbol{\varepsilon}}^{pl}, \dot{\alpha})}{\partial \boldsymbol{\tau}} &= -\dot{\boldsymbol{\varepsilon}}^{pl} + \dot{\gamma}\frac{\partial \Phi}{\partial \boldsymbol{\tau}} = 0 \\
\frac{\partial \mathcal{L}^{pl}(\boldsymbol{\tau}, p, \dot{\gamma}; \dot{\boldsymbol{\varepsilon}}^{pl}, \dot{\alpha})}{\partial p} &= \dot{\alpha} + \dot{\gamma}\frac{\partial \Phi}{\partial p} = 0
\end{aligned} \tag{6}$$

with the standard KUHN–TUCKER and the consistency condition

$$\dot{\gamma} \geq 0, \qquad \Phi(\boldsymbol{\sigma}, \phi) \leq 0, \qquad \dot{\gamma}\, \Phi(\boldsymbol{\sigma}, \phi) = 0, \qquad \dot{\Phi}\dot{\gamma} = 0 \tag{7}$$

3 VARIATIONAL FORMULATION

In the following section we want to present the variational inequalities for classical Prandtl–Reuss plasticity, DUVAUT & LIONS [9], JOHNSON [13] and HAN & REDDY [11]. Introducing the following function spaces:

$$\begin{aligned}
\text{where:}\quad & \mathcal{H} := L^2(\Omega, \mathbb{R}^n), && \mathcal{S} := L^2(\Omega, \mathbb{R}^{(n\times n)}) \\
& && \mathcal{S}^{div} := \{\boldsymbol{\tau} \in \mathcal{S}, \mathrm{div}\, \boldsymbol{\tau} \in \mathcal{H}\} \\
& \mathcal{V} := \{\boldsymbol{v} \in \mathcal{H}^1_0(\Omega)\}, && \mathcal{E} = \Pi\mathcal{S} := \{\boldsymbol{\tau} \in \mathcal{S},\ \Phi(\boldsymbol{\tau}, p) \leq 0\}
\end{aligned} \tag{8}$$

Now we use the *Postulate of Maximum Plastic Dissipation* and introduce the velocities $\dot{\boldsymbol{u}} = \frac{\partial \boldsymbol{u}}{\partial t}$. After the integration over the space domain Ω we add the weak form of the equilibrium equation and end up in the well known *dual–mixed form*

$$(\boldsymbol{C}^{-1} : \dot{\boldsymbol{\sigma}}, \boldsymbol{\tau} - \boldsymbol{\sigma}) + (\dot{\boldsymbol{u}}, \mathrm{div}\, \boldsymbol{\tau} - \mathrm{div}\, \boldsymbol{\sigma}) - (\mathrm{div}\, \boldsymbol{\sigma}, \boldsymbol{v}) \geq (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall \{\boldsymbol{v}, \boldsymbol{\tau}\} \in \mathcal{H} \times \mathcal{E}^{div}_0 \tag{9}$$

and after partial integration we get the *primal–mixed form*

$$(\boldsymbol{C}^{-1} : \dot{\boldsymbol{\sigma}}, \boldsymbol{\tau} - \boldsymbol{\sigma}) - (\dot{\boldsymbol{\varepsilon}}, \boldsymbol{\tau} - \boldsymbol{\sigma}) + a(\boldsymbol{u}, \boldsymbol{v}) \geq (\boldsymbol{f}, \boldsymbol{v}) + (\boldsymbol{g}, \boldsymbol{v})_{\Gamma_N} \qquad \forall \{\boldsymbol{v}, \boldsymbol{\tau}\} \in \mathcal{V} \times \mathcal{E} \quad (10)$$

The bilinear form $a(..,..)$ is associated with the dual variables and $\boldsymbol{C}$ is the 4th order elasticity tensor:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{v}) \tag{11}$$

Next we involve some viscoplastic regularization, Simo & Hughes [20]. The viscoplastic regularization of the rate–independent Prandtl–Reuss equations has the effect, that the governing equations remain hyperbolic in the case of negative hardening. This is essential for softening materials to get mesh independent numerical solutions.
First we consider the constrained minimization problem eq. (6) in which the solution is $\boldsymbol{\sigma} \in \mathcal{E}$.

$$\min_{\boldsymbol{\tau} \in \mathcal{E}_\sigma} \{-\mathcal{D}^{pl}(\boldsymbol{\tau}, p; \dot{\boldsymbol{\varepsilon}}^{pl}, \dot{\alpha})\} \tag{12}$$

$$\text{with} \qquad \mathcal{D}^{vp}_\eta = \boldsymbol{\tau} : \dot{\boldsymbol{\varepsilon}}^{vp} - p\,\dot{\alpha} \tag{13}$$

Associated with this problem is the unconstrained minimisation problem with the solution $\boldsymbol{\tau} \in \mathcal{S}$

$$\min_{\boldsymbol{\tau} \in \mathcal{S}_\sigma} \{-\mathcal{D}^{vp}_\eta(\boldsymbol{\tau}, p; \dot{\boldsymbol{\varepsilon}}^{pl}, \dot{\alpha})\} \tag{14}$$

$$\text{with} \qquad \mathcal{D}^{vp}_\eta = \boldsymbol{\tau} : \dot{\boldsymbol{\varepsilon}}^{vp} - p\,\dot{\alpha} + \frac{1}{\eta}(\Phi(\boldsymbol{\tau}, p)) \tag{15}$$

$\eta \in [0, \infty]$ is a penalty parameter. The conditions to involve such a penalty regularisation technique are described for example in Luenberger [15]. The constraint, here the yield function Φ, has to be a C^1 continuous function.

$$\frac{\partial \mathcal{D}^{vp}_\eta}{\partial p} = -\dot{\alpha} + \frac{1}{\eta}\langle\Phi\rangle\frac{\partial \Phi}{\partial p} \stackrel{!}{=} 0 \tag{16}$$

$$\frac{\partial \mathcal{D}^{vp}_\eta}{\partial \boldsymbol{\tau}} = \dot{\boldsymbol{\varepsilon}}^{vp} + \frac{1}{\eta}\langle\Phi\rangle\frac{\partial \Phi}{\partial \boldsymbol{\tau}} \stackrel{!}{=} 0 \tag{17}$$

$$\Rightarrow \quad \dot{\boldsymbol{\varepsilon}}^{vp} = -\frac{1}{\eta}\langle\Phi\rangle\frac{\partial \Phi}{\partial \boldsymbol{\tau}} = \frac{1}{\eta}\left\{\mathrm{dev}\boldsymbol{\sigma} - \frac{\sigma_y}{\|\mathrm{dev}\boldsymbol{\sigma}\|}\mathrm{dev}\boldsymbol{\sigma}\right\} \stackrel{!}{=} \frac{\mathrm{dev}\boldsymbol{\sigma} - \Pi\mathrm{dev}\boldsymbol{\sigma}}{\eta} \tag{18}$$

From a more mechanical point of view the obtained evolution equations follow the classical Perzyna–viscoplasticity model, [18]. The physical interpretation of the penalty (fluidity) factor η is the relaxation time which is described by $\eta/2\mu$. The projection Π is given in eq (22). It is apparent that we obtain the Prandtl–Reuss solution for $\eta \to 0$. The proof of this result is straightforward and can also be found in Luenberger [15].

Using the weak form of eq. (2a) and inserting the regularized (viscoplastic) strains eq. (18)

$$(\dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}),\boldsymbol{\tau}) = \frac{1}{\eta}(\mathrm{dev}\boldsymbol{\sigma} - \Pi\ \mathrm{dev}\boldsymbol{\sigma},\boldsymbol{\tau}) + (\boldsymbol{C}^{-1} : \dot{\boldsymbol{\sigma}},\boldsymbol{\tau}) \quad \forall\{\boldsymbol{v},\boldsymbol{\tau}\} \in \mathcal{V}\times\mathcal{S} \tag{19}$$

and adding the weak form of the equilibrium equation we end up in the *regularized primal mixed form* as a (nonlinear) variational equality, Rannacher & Suttmeier [19], Duvaut & Lions [9] which is the starting point for developing our error estimators.

$$(\boldsymbol{C}^{-1} : \dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}),\boldsymbol{\tau}) + \frac{1}{\eta}(\mathrm{dev}\boldsymbol{\sigma} - \Pi\ \mathrm{dev}\boldsymbol{\sigma},\boldsymbol{\tau}) + a(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) + (\boldsymbol{g},\boldsymbol{v})_{\Gamma_N} \quad \forall\{\boldsymbol{v},\boldsymbol{\tau}\} \in \mathcal{V}\times\mathcal{S} \tag{20}$$

Eliminating the stresses $\boldsymbol{\tau}$ from eq. (20) we obtain the *primal* form as a nonlinear variational equation. This is the governing weak form for the present pure displacement approach.

$$(\Pi\ (\boldsymbol{C} : \boldsymbol{\varepsilon}(\boldsymbol{u})),\boldsymbol{\varepsilon}(\boldsymbol{v})) = (\boldsymbol{f},\boldsymbol{v}) + (\boldsymbol{g},\boldsymbol{v})_{\Gamma_N} \qquad \forall \boldsymbol{v} \in \mathcal{V} \tag{21}$$

with the projection as a further equation

$$\Pi\ \mathrm{dev}\boldsymbol{\tau} := \begin{cases} \mathrm{dev}\boldsymbol{\tau} & \|\boldsymbol{\tau}\| \leq \sigma_y \\ -\phi(\alpha)\dfrac{\mathrm{dev}\boldsymbol{\tau}}{\|\mathrm{dev}\boldsymbol{\tau}\|} & \|\boldsymbol{\tau}\| > \sigma_y \end{cases} \tag{22}$$

4 DISCRETIZATION SCHEMES

Choosing some subintervalls $I_k := (T_{k-1}, T_k)$ with the length $|I_k| = \Delta T$ and integrate the regularised problem (20) over one time-step I_k we obtain

$$\int_{T_{k-1}}^{T_k}(\boldsymbol{C}^{-1} : \dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\varepsilon}}(\boldsymbol{u}),\boldsymbol{\tau}) + \int_{T_{k-1}}^{T_k}\frac{1}{\eta}(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma},\boldsymbol{\tau}) + \int_{T_{k-1}}^{T_k} a(\boldsymbol{u},\boldsymbol{v}) = \int_{T_{k-1}}^{T_k}(\boldsymbol{f},\boldsymbol{v}) + \int_{T_{k-1}}^{T_k}(\boldsymbol{g},\boldsymbol{v})_{\Gamma_N} \tag{23}$$

and applying the backward Euler–integration scheme which reads

$$\boldsymbol{x}^{T_k} = \boldsymbol{x}^{T_{k-1}} + \int_{T_{k-1}}^{T_k}\boldsymbol{f}(t) \qquad \text{and} \qquad \int_{T_{k-1}}^{T_k}\boldsymbol{f}(t) = \Delta T\boldsymbol{f}(T^k) \tag{24}$$

This is leading to a semi–discrete form of our *regularised primal mixed form* for an actual time step

$$\begin{aligned} & (\boldsymbol{C}^{-1} : \boldsymbol{\sigma}^{T_k} - \boldsymbol{C}^{-1} : \boldsymbol{\sigma}^{T_{k-1}},\boldsymbol{\tau}) - \Delta T(\boldsymbol{\varepsilon}^{T_k},\boldsymbol{\tau}) \\ + \ & \frac{\Delta T}{\eta}(\boldsymbol{\sigma}^{T_k} - \Pi\boldsymbol{\sigma}^{T_k},\boldsymbol{\tau}) + \Delta T(\boldsymbol{\sigma}^{T_k},\boldsymbol{\varepsilon}) \\ = \ & \Delta T(\boldsymbol{f}^{T_k},\boldsymbol{v}) + \Delta T(\boldsymbol{g}^{T_k},\boldsymbol{v})_{\Gamma_N} \end{aligned} \tag{25}$$

Instead of using the simple backward–Euler scheme to integrate eq. (20) we also can apply (embedded) RUNGE–KUTTA integration schemes. For this standard techniques from ODEs can be used to estimate the time–stepping error, DIEBELS ET AL. [8].

Another possibility is using a GALERKIN formulation in space and time. For this the coupled space–time error can be captured.

5 ERROR ESTIMATION FOR THE REGULARISED PROBLEM

Next we will continue estimating the pure discretisation error in space domain. Due to the primal or displacement approach the weak form of our problem, which can be interpreted as a semi–discrete equation (continuous in space, discretised in time domain) is given by

$$(\Pi\ (\boldsymbol{C} : \boldsymbol{\varepsilon}^{T_k}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v})) = (\boldsymbol{f}^{T_k}, \boldsymbol{v}) + (\boldsymbol{g}^{T_k}, \boldsymbol{v})_{\Gamma_n} \qquad \forall\ \boldsymbol{v} \in \mathcal{V} \tag{26}$$

where we could define a bilinear form

$$a(\boldsymbol{u}, \boldsymbol{v}) := (\Pi\ (\boldsymbol{C} : \boldsymbol{\varepsilon}^{T_k}(\boldsymbol{u})), \boldsymbol{\varepsilon}(\boldsymbol{v})) \tag{27}$$

and a linear form

$$l(\boldsymbol{v}) := (\boldsymbol{f}^{T_k}, \boldsymbol{v}) + (\boldsymbol{g}^{T_k}, \boldsymbol{v})_{\Gamma_n} \tag{28}$$

with

$$a(\boldsymbol{u}, \boldsymbol{v}) = l(\boldsymbol{v}) \qquad \forall\ \boldsymbol{v} \in \mathcal{V} \tag{29}$$

For this we can formulate our finite element problem: find $\boldsymbol{u}_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h) = l(\boldsymbol{v}_h) \qquad \forall \{\boldsymbol{v}_h, \boldsymbol{u}_h\} \in \mathcal{V}_h \subset \mathcal{V} \tag{30}$$

Introducing the error in the displacements

$$\boldsymbol{e} := \boldsymbol{u} - \boldsymbol{u}_h \tag{31}$$

and subtracting eq. (29) from eq. (30) gives us a GALERKIN–orthogonality in the tangential space at a discrete point T_k in the time domain $[0, T]$. Here we would like to mention the fundamental paper of RANNACHER & SUTTMEIER [19], in which this orthogonality condition is called a nonlinear GALERKIN–orthogonality. Using this GALERKIN–orthogonality

$$a(\boldsymbol{e}, \boldsymbol{v}_h) = 0 \tag{32}$$

and the definition of the error, we get the weak form of the error equation

$$\begin{aligned} a(\boldsymbol{e}, \boldsymbol{v}) &= a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) = l(\boldsymbol{v}) - a(\boldsymbol{u}_h, \boldsymbol{v}) \\ &= \sum_{T \in \mathcal{T}_h} \int_T \mathrm{Res}_T(\boldsymbol{u}_h)\ \boldsymbol{v} + \sum_{E \in \mathcal{E}_h} \int_E \mathrm{Res}_E(\boldsymbol{u}_h)\ \boldsymbol{v} \end{aligned} \tag{33}$$

with the definition of the element inner error and the jump terms. The subscripts $_{i,j}$ stand for two neighbouring elements joining the same element edge

$$\begin{aligned} \mathrm{Res}_T(\boldsymbol{u}_h) &:= \mathrm{div}\boldsymbol{\sigma}(\boldsymbol{u}_h) + \boldsymbol{f} & T \in \mathcal{T}_h \\ \mathrm{Res}_E(\boldsymbol{u}_h) &:= \begin{cases} -\dfrac{1}{2}(\boldsymbol{n}_i \cdot \boldsymbol{\sigma}_i(\boldsymbol{u}_h) - \boldsymbol{n}_j \cdot \boldsymbol{\sigma}_j(\boldsymbol{u}_h)) & E \in \mathcal{E}_{h,\Omega} \\ \boldsymbol{g} - \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{u}_h) & E \in \mathcal{E}_{h,\Gamma_N} \\ \boldsymbol{0} & E \in \mathcal{E}_{h,\Gamma_D} \end{cases} \end{aligned} \tag{34}$$

where the jump terms are distributed by the factor 0.5 to the elements $_{i,j}$.

Of course, it is possible to discretize eq. (33) and solve this problem by higher order finite elements or on a finer grid which is leading to a kind of overkill solution. But instead of discretising and solving this problem we want to estimate the discretization error.

Error estimators based on this kind of weak form of the error are leading to global explicit or implicit residual based error estimators. The underlying optimization problem for the finite element mesh can be formulated as: *finding a mesh with uniform error distribution in the whole domain* Ω , Babuška & Rheinboldt [2], Bank & Weiser [3], Ladèveze & Leguillon [14].

The goal of the error estimator we want to present further on differs totally from this optimization problem. The new optimization problem is called: *finding a optimal finite element mesh for a minimised error of a local quantity*, Tottenham [21], Becker & Rannacher [4], and Cirak & Ramm [6]. The error–controlled local quantities can be e.g. a displacement field in a small area, stress concentration close to the tip of a crack, calculated by contour integrals, maximum values of plastic strains etc.

To estimate such local quantities we have to introduce a dual problem. This is a well known technique in the field of the *a priori* analysis, Nitsche & Schatz [16], and was developed for the *a posteriori* analysis by the group of Johnson. The right hand side of this problem has to be chosen in a way that it is dual in the energy sense to the error–controlled quantities. So the solution of the dual problem acts like a filter function or in engineering words influence function for the residuals of the primal problem.

In contrast to the work of Johnson et al. and similar to the work of Rannacher et al. the dual problem is discretised and solved numerically, often on the same mesh, as the underlying primal problem.

We introduce a (linearised) dual problem:

$$\begin{aligned} -\mathrm{div}\,\boldsymbol{\sigma}(\boldsymbol{G}) &= \delta(\bar{\boldsymbol{x}}) && \text{in } \Omega \\ \boldsymbol{G} &= \boldsymbol{0} && \text{on } \Gamma_D \\ \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{G}) &= \boldsymbol{0} && \text{on } \Gamma_N \end{aligned} \tag{35}$$

	global error control	local error control
explicit residual based	$a(\boldsymbol{e}, \boldsymbol{v}) = (Res(\boldsymbol{u}_h), \boldsymbol{v} - \boldsymbol{v}_h)$	$a(\boldsymbol{e}, \delta(\bar{\boldsymbol{x}})) = (Res(\boldsymbol{u}_h), \boldsymbol{G} - \boldsymbol{G}_h)$
implicit residual based	$a(\boldsymbol{e}, \boldsymbol{v}) = (Res(\boldsymbol{u}_h), \boldsymbol{v} - \boldsymbol{v}_h)$ + local BVP for primal problem	$a(\boldsymbol{e}, \delta(\bar{\boldsymbol{x}})) = (Res(\boldsymbol{u}_h), \boldsymbol{G} - \boldsymbol{G}_h)$ + local BVP for dual problem
gradient type	$\lVert \boldsymbol{e} \rVert^2 \approx$ $(\boldsymbol{\sigma}^\star(\boldsymbol{u}_h) - \boldsymbol{\sigma}(\boldsymbol{u}_h), \boldsymbol{\epsilon}^\star(\boldsymbol{u}_h) - \boldsymbol{\epsilon}(\boldsymbol{u}_h))$	$a(\boldsymbol{e}, \delta(\bar{\boldsymbol{x}})) \approx$ $(\boldsymbol{\sigma}^\star(\boldsymbol{u}_h) - \boldsymbol{\sigma}(\boldsymbol{u}_h), \boldsymbol{\epsilon}^\star(\boldsymbol{G}_h) - \boldsymbol{\epsilon}(\boldsymbol{G}_h))$

Table 1: classification of local and global error estimators

The solution of the dual problem is $\boldsymbol{G}$ and the right hand side $\delta(\bar{\boldsymbol{x}})$ is a regularisation of the Dirac–Delta function $\delta(\boldsymbol{x})$ so that the corresponding energy is bounded. If we use the reciprocal theorem of Betti & Maxwell

$$(\boldsymbol{e}, \delta(\bar{\boldsymbol{x}})) = -(\boldsymbol{e}, \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{G})) = (\boldsymbol{C} : \nabla \boldsymbol{e}, \nabla \boldsymbol{G}) = (\mathrm{Res}, \boldsymbol{G}) \tag{36}$$

we end up in the local weak form of the error equation

$$a(\boldsymbol{e}, \delta(\bar{\boldsymbol{x}})) = |\boldsymbol{e}(\bar{\boldsymbol{x}})| = \sum_{T \in \mathcal{T}_h} \int_T \mathrm{Res}_T(\boldsymbol{u}_h)\, \boldsymbol{G} + \sum_{E \in \mathcal{E}_h} \int_E \mathrm{Res}_E(\boldsymbol{u}_h)\, \boldsymbol{G} \tag{37}$$

Using the Galerkin–orthogonality we get

$$|\boldsymbol{e}(\bar{\boldsymbol{x}})| = \sum_{T \in \mathcal{T}_h} \int_T \mathrm{Res}_T(\boldsymbol{u}_h)\, (\boldsymbol{G} - \boldsymbol{G}_h) + \sum_{E \in \mathcal{E}_h} \int_E \mathrm{Res}_E(\boldsymbol{u}_h)\, (\boldsymbol{G} - \boldsymbol{G}_h) \tag{38}$$

To classify the possibilities we extend the definition of global error estimators due to Verfuerth [22] and Ainsworth & Oden [1] to local error estimators, Table 1.

Applying the Hölder–inequality for integrals the error of the local variable is estimated

$$|\boldsymbol{e}(\bar{\boldsymbol{x}})| \leq \sum_{T \in \mathcal{T}_h} \lVert \mathrm{Res}_T(\boldsymbol{u}_h) \rVert \, \lVert \boldsymbol{G} - \boldsymbol{G}_h \rVert + \sum_{E \in \mathcal{E}_h} \lVert \mathrm{Res}_E(\boldsymbol{u}_h) \rVert \, \lVert \boldsymbol{G} - \boldsymbol{G}_h \rVert \tag{39}$$

Here we want to remark that we have several possibilities to estimate the error of the dual problem. In this approach we solve the error of the dual problem solving 2^{nd} order difference quotient.

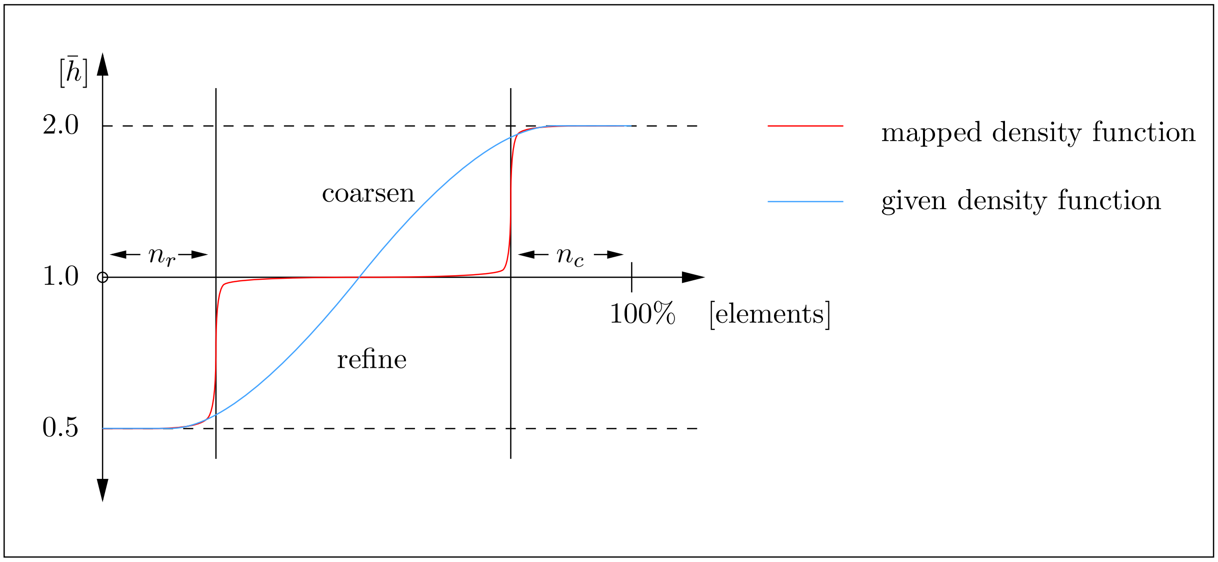


Figure 1: fixed fraction method for advancing front mesh generation

Standard interpolation operators

$$|e(\bar{\boldsymbol{x}})| \leq \sum_{T\in\mathcal{T}_h} \|\mathrm{Res}_T(\boldsymbol{u}_h)\| \; \|\boldsymbol{G}-\Pi_h\boldsymbol{G}\| + \sum_{E\in\mathcal{E}_h} \|\mathrm{Res}_E(\boldsymbol{u}_h)\| \; \|\boldsymbol{G}-\Pi_h\boldsymbol{G}\| \tag{40}$$

are introduced, Braess [5]

$$\|\boldsymbol{G}-\Pi_h\boldsymbol{G}\|_{0,T} \leq C_{i,1} h_T^2 |\boldsymbol{G}|_{2,T} \tag{41}$$

$$\|\boldsymbol{G}-\Pi_h\boldsymbol{G}\|_{0,E} \leq C_{i,2} h_T^{\frac{3}{2}} |\boldsymbol{G}|_{2,T} \tag{42}$$

with interpolation constants $C_{i,1}$ and $C_{i,2}$ and a characteristic element length h_T. Inserting the interpolation estimates in the eq. (40) we obtain

$$|\boldsymbol{e}(\bar{\boldsymbol{x}})| \leq \bar{\eta} := C_{i,3} \left\{ \sum_{T\in\mathcal{T}_h} h_T^2 \|\mathrm{Res}_T(\boldsymbol{u}_h)\| \; |\boldsymbol{G}|_{2,T} + \sum_{E\in\mathcal{E}_h} h_T^{\frac{3}{2}} \|\mathrm{Res}_E(\boldsymbol{u}_h)\| \; |\boldsymbol{G}|_{2,T} \right\} \tag{43}$$

Assuming for simplicity $C_i = 1$ all terms of the right side of eq. (43) are known and we are able to calculate the element–wise error. It needs to be said that we get sharper error bounds if we do not apply the Hölder inequality which is roughly speaking a tough estimation.

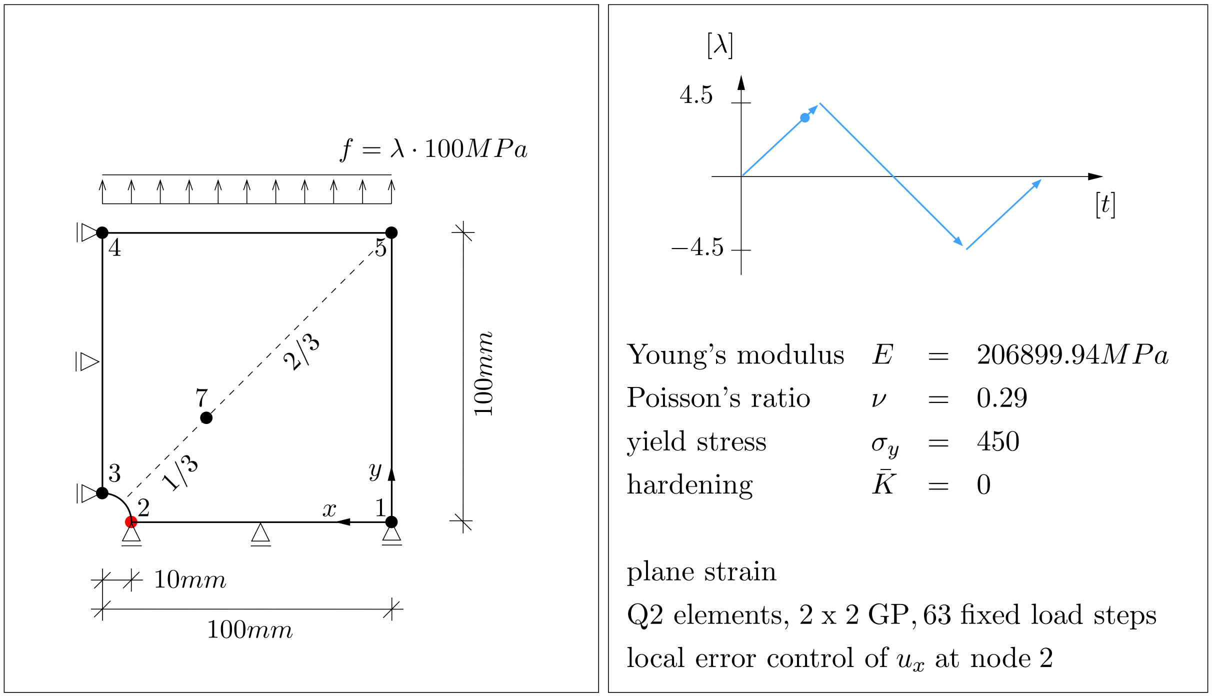


Figure 2: benchmark problem, stretched plane strain plate with a hole

6 MESH REFINEMENT STRATEGY

The discretised linearised boundary value problem eq. (26) is solved in an adaptive process. After solving the nonlinear problem by using a Newton–Raphson iteration the *a posteriori* error indicator eq. (43) is calculated at the new equilibrium point. For this the linearised dual problem based on the actual properties of the nonlinear problem has to be evaluated. By saving the triangularised stiffness matrix for the equilibrium state of the primal problem we only have to perform one additional backward substitution for the dual problem. To estimate the error of the dual problem based on the discrete solution $\boldsymbol{G}_h$ we have several possibilities, Table 1. Here we estimate the error explicitly by calculating 2^{nd} order difference quotient. By mapping the local error indicator $\bar{\eta}$ onto a density function which is based on a fixed fraction method we can apply a mesh generation based e.g. on the advancing front technique, Figure 1. Here $\bar{h}$ is a given characteristic element length, e.g. the diameter of an element. The advantage of this approach is that we can coarsen and refine a mesh with a given number of elements. That is important in the case of calculating limit points. For these problems it is important to refine/coarsen only at a given number of elements to reach convergence in the global Newton iteration. If too many elements are refined/coarsened the mapping error could be too large and it is possible to lose convergence. For the PRANDTL–REUSS plasticity as a typical path–dependent problem we have to transform the nodal variables $\boldsymbol{u}_h$ and the set of internal variables $\boldsymbol{q} = (\boldsymbol{\varepsilon}^{pl}, \alpha)$ from the old mesh to the new mesh. The internal variables are calculated at the Gauss points and are in our displacement approach C^{-1} continuous. By mapping the internal variables to the Gauss points and applying a simple nodal averaging technique we get "improved" internal variables which are in C^0. Then we map the internal variables to the Gauss points of the new mesh. After mapping the nodal and internal variables to the new mesh, we have to control the global equilibrium state. This is done by minimising the residual forces in a further global Newton–Raphson iteration step.

7 NUMERICAL EXAMPLE

This numerical example is a standard benchmark example [12]. A two dimensional square plate with a hole is subjected to a constant boundary traction. Here we assume elastic–perfectly plastic material without any hardening under plane strain conditions. Because of the symmetry conditions we consider only a quarter of the problem in the numerical calculation . We apply a cyclic loading process described in Figure 2. To test our local error approach we are now interested in controlling the error of the horizontal displacement $\boldsymbol{u}_x$ at the point 2. In our adaptive finite element calculation our final mesh consists on 2934 $Q2$ elements with 17772 DOF and we apply 63 fixed load steps. No adaptive time–stepping scheme was applied.

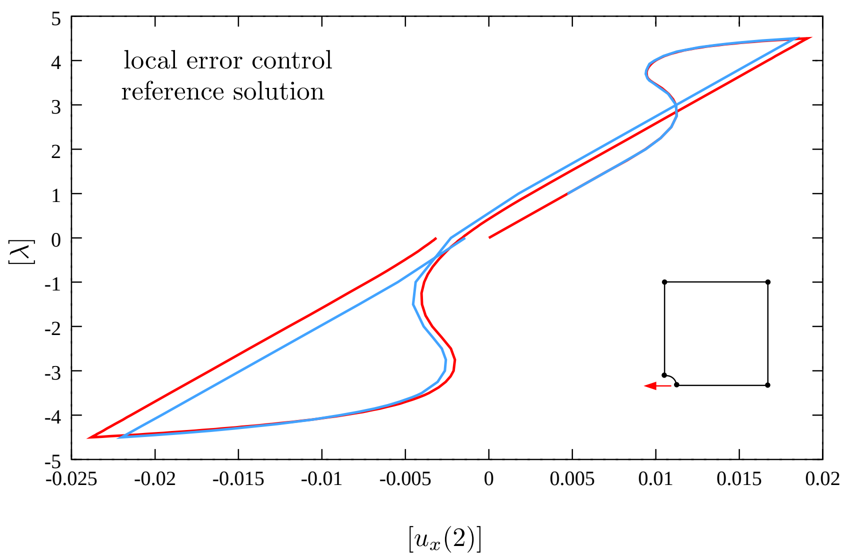


Figure 3: load displacement diagram

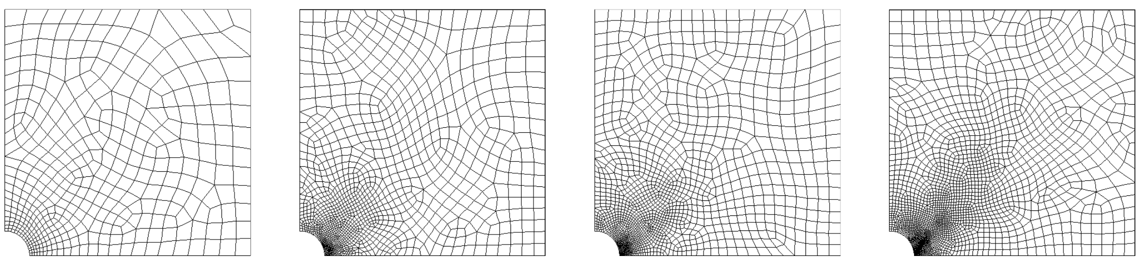
As we can see in the given load displacement diagram Figure 3 we can achieve a remarkably good result compared to a solution based on a reference solution. The reference solution was calculated with 65536 $Q2/P1$ elements with 197633 DOF and 205 load steps by

Wieners [12].

8 CONCLUSIONS AND FUTURE WORK

The local or in other words goal oriented error estimator presented here could be applied very efficiently to control local error quantities. In the framework presented here it could be generally applied for plastic or viscoplastic materials. As we have pointed out the additional numerical cost in a nonlinear analysis to solve the dual problem numerically is negligible.

We want to remark, that the present error indicator eq. (43) is similar to the error indicator proposed by Suttmeier & Rannacher [19]. The difference is the underlying adaptive finite element scheme for viscoplastic problems, which will be shown by some numerical examples with softening behaviour in the presentation. Because of the loss of ellipticity in the underlying differential equations we need to introduce a regularisation technique, e.g. the viscous regularisation mentioned above.



2536 DOF, 410 ele. $\lambda = 1.50$ | 8042 DOF, 1319 ele. $\lambda = 4.10$ | 9736 DOF, 1600 ele. $\lambda = 4.40$ | 17772 DOF, 2934 ele. $\lambda = -4.00$

Figure 4: sequence of refined meshes

In contrast to the local error indicator for Prandtl–Reuss plasticity proposed by Cirak & Ramm [7] we do not use any smoothing procedures to evaluate the error of the primal and dual solution in which a global energy norm error estimator is evaluated twice. In the present approach the residuals of the primal problem can be calculated element–wise as an explicit result of the primal solution. Only for the second order derivatives which are needed to estimate the error of the dual problem a small patch–wise problem for the element T_i and the adjacent elements T_k has to be solved.

Another adressed topic is to compare the different approaches for estimating the error of the dual problem. For example the error of the dual part may be estimated by solving local Neumann problems, Ohnimus et at. [17]. Through this we hope to improve the local error estimator introducing sharper error bounds.

References

[1] Ainsworth, M. & J. Oden [1997]. 'A posteriori error estimation in finite element analysis.' *Comput. Methods Appl. Mech. Engrg.*, **142**, pp. 1–88.

[2] Babuška, I. & W. Rheinboldt [1978]. 'Error estimates for adaptive finite element computations.' *SIAM J. Numer. Anal.*, **15**(4), pp. 736–754.

[3] Bank, R. & A. Weiser [1985]. 'Some a posteriori error estimators for elliptic partial differential equations.' *Math. Comp.*, **44**, pp. 283–301.

[4] Becker, R. & R. Rannacher [1996]. 'A feed–back approach to error control in finite element methods.' *East-West J. Numer. Math.*, **4**, pp. 237–264.

[5] Braess, D. [1997]. *Finite element methods*. Cambridge University Press.

[6] Cirak, F. & E. Ramm [1998]. 'A-posteriori error estimation and adaptivity for linear elasticity using the reciprocal theorem.' *Comp. Meth. in Appl. Mech. Eng.*, **156**, pp. 351–362.

[7] Cirak, F. & E. Ramm [2000]. 'A *posteriori* error estimation and adaptivity for elastoplasticity using the reciprocal theorem.' *Int. J. Num. Meth. Eng.*, **47**, pp. 379–393.

[8] Diebels, S., P. Ellsiepen & W. Ehlers [1999]. 'Error–controlled Runge–Kutta time integration of a viscoplastic hybrid two–phase model.' *Technische Mechanik*, **19**, pp. 19–29.

[9] Duvaut, G. & J. Lions [1976]. *Inequalities in mechanics and physics*. A series of comprehensive studies in mathematics. Springer–Verlag, Berlin.

[10] Eriksson, K., D. Estep, P. Hansbo & C. Johnson [1995]. 'Introduction to adaptive methods for differential equations.' *Acta Numerica*, pp. 105–158.

[11] Han, W. & B. Reddy [1999]. *Plasticity*, volume 9 of *Interdisciplinary Applied Mathematics*. Springer–Verlag, Berlin.

[12] Homepage of DFG–Paketantrag. *Adaptive finite element methods in applied mechanics*. http://www.ibnm.uni-hannover.de/Forschung/Paketantrag/Bench–mark/benchmark.html.

[13] Johnson, C. [1976]. 'Existence theorems for plasticity problems.' *J. Math. pures et appl.*, **55**, pp. 431–444.

[14] Ladèveze, P. & D. Leguillon [1983]. 'Error estimate procedure in the finite element method and applications.' *SIAM J. Numer. Anal.*, **20**(3), pp. 485–509.

[15] LUENBERGER, D. G. [1984]. *Linear and Nonlinear Programming*. Addison-Wesly Publ. Co.

[16] NITSCHE, J. & A. SCHATZ [1974]. 'Interior estimates for Ritz–Galerkin methods.' *Math. Comp.*, **28**(128), pp. 937–958.

[17] OHNIMUS, S., E. STEIN & E. WALHORN [2000]. 'Local error estimates of FEM for displacements and stresses in linear elasticity by solving local Neumann problems.' preprint.

[18] PERZYNA, P. [1971]. *Thermodynamic Theory of Viscoplasticity*, volume 11 of *Advances in Applied Mechanics*. Academic Press, New York.

[19] RANNACHER, R. & F.-T. SUTTMEIER [1999]. 'A posteriori error estimation and mesh adaption for finite element models in elasto–plasticity.' *Comput. Methods Appl. Mech. Engrg.*, **176**, pp. 333–361.

[20] SIMO, J. & T. HUGHES [1997]. *Computational Inelasticity*, volume 7 of *Interdisciplinary Applied Mathematics*. Springer–Verlag, Berlin.

[21] TOTTENHAM, H. [1970]. 'Basic principles.' In *Finite Element Techniques in Structural Mechanics*, edited by H. Tottenham & C. Brebbia. Southampton University Press, Southampton.

[22] VERFÜRTH, R. [1995]. *A review of a posteriori error estimation and adaptive mesh–refinement techniques*. Wiley & Sons, B. G. Teubner.